NAME:

Solutions to Math 150 Practice Exam 2.1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. Compute $\frac{dy}{dx}$ where $y = (\sin^2 x + 1)^4$ [10 pts]

Solution: $\frac{dy}{dx} = 4(\sin^2 x + 1)^3 2 \sin x \cos x = 8(\sin^2 x + 1)^3 \sin x \cos x$

Remark: Using the identity $2 \sin x \cos x = \sin 2x$ we can also write the solution as

$$\frac{dy}{dx} = 4(\sin^2 x + 1)^3 \sin 2x$$

2. Compute $\frac{dy}{dx}$ where $y = \left(\frac{x-1}{x+1}\right)^8$ [10 pts]

Solution: $\frac{dy}{dx} = 8\left(\frac{x-1}{x+1}\right)^7 \frac{(x+1)-(x-1)}{(x+1)^2} = 8\left(\frac{x-1}{x+1}\right)^7 \frac{2}{(x+1)^2} = 16\left(\frac{x-1}{x+1}\right)^7 \frac{1}{(x+1)^2}$

3. Calculate $\frac{dy}{dx}$ implicitly from the equation $\sin xy = x + y$ [10 pts]

Solution: $\frac{d}{dx}(\sin xy) = \frac{d}{dx}(x + y)$. Therefore

$$\cos xy \left(y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

Solving this equation for $\frac{dy}{dx}$ we obtain

$$\frac{dy}{dx} = \frac{1 - y\cos xy}{x\cos xy - 1}$$

4. Find an equation of the tangent line to the curve $x^4 + y^4 = 2$ at the point $(1, -1)$ [10 pts]

Solution: $\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(2)$. Therefore $4x^3 + 4y^3\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = \frac{-x^3}{y^3}$. In particular,

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-(-1)^3}{1^3} = 1$$

Hence the equation of the tangent line is

$$y + 1 = x - 1$$

5. Calculate $\frac{dy}{dx}$ implicitly from the equation $\sqrt{x^4 + y^2} = 5x + 2y^3$

[10 pts]

Solution: $\frac{d}{dx}\left(\sqrt{x^4 + y^2}\right) = 5 + 6y^2\frac{dy}{dx}$. Hence

$$\frac{4x^3 + 2y\frac{dy}{dx}}{2\sqrt{x^4 + y^2}} = 5 + 6y^2\frac{dy}{dx}$$

Simplifying and multiplying both sides of the equation by $\sqrt{x^4 + y^2}$ we get

$$2x^3 + y\frac{dy}{dx} = 5\sqrt{x^4 + y^2} + 6y^2\sqrt{x^4 + y^2}\frac{dy}{dx}$$

Therefore,

$$\frac{dy}{dx} = \frac{5\sqrt{x^4 + y^2} - 2x^3}{y - 6y^2\sqrt{x^4 + y^2}}$$

6. A bug is moving along the parabola $y = x^2$. At what point on the parabola are the x- and y-coordinates changing at the same rate? [10 pts]

Solution: The x- and y-coordinates changing at the same rate if and only if $\frac{dx}{dt} = \frac{dy}{dt}$. So we must find all points where this equation is true. Since $y = x^2$, we have

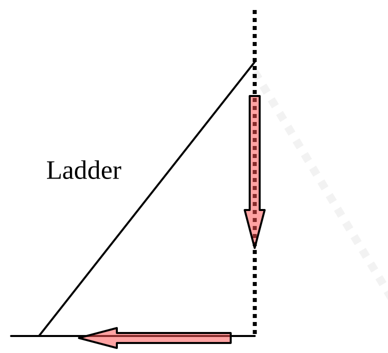
$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

Therefore, given that $\frac{dx}{dt} = \frac{dy}{dt}$, we obtain

$$2x = 1 \text{ or } x = 1/2$$

Hence the desired point is $\left(\frac{1}{2}, \frac{1}{4}\right)$.

7. A 13-foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/s. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall? [10 pts]



Solution: Let $x(t)$ be the distance of the foot of the ladder from the wall and let $y(t)$ be the distance from the head of the ladder to the floor. Then $x^2(t) + y^2(t) = 169$ at every moment t. We have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

When $x = 5$, $y = 12$ and since $\frac{dx}{dt} = 0.5$, we have

$$5 \cdot 0.5 + 12\frac{dy}{dx} = 0$$

Thus, $\frac{dy}{dx} = -\frac{5}{24}$ ft/s.

8. What two positive real numbers whose product is 50 have the smallest possible sum? [10 pts]

Solution: Let x and y be the two numbers whose product is 50. Then $y = \frac{50}{x}$ and we would like to minimize the sum $S(x) = x + \frac{50}{x}$. Setting $S'(x) = 1 - \frac{50}{x^2} = 0$ we see that $x^2 = 50$ corresponds to a critical point. Using the first derivative test, conclude that $x = 5\sqrt{2}$ corresponds to a local minimum. This, in fact, must be the absolute minimum (why?).

Note that the Extreme Value Theorem does not apply.

9. Let $f(x) = \sqrt{x}$. Find all numbers c that satisfy the statement of the Mean Value Theorem in the interval $[1, 4]$. Be sure to explain why the Mean Value Theorem applies to the given function. [10 pts]

Solution: Observe that the hypothesis of the Mean-Value theorem is satisfied for the function $f(x) = \sqrt{x}$ over the interval$[1, 4]$.Now

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{3} = \frac{1}{3} = f'(c) = \frac{1}{2\sqrt{c}}$$

Hence, $c = \frac{9}{4}$ is the only solution that satisfies the Mean-Value Theorem

10. A state patrol officer saw a car start from rest at a highway on-ram. She radioed ahead to a patrol officer 30 mi along the highway. When the car reached the location of the second officer 28 min later, it was clocked going 60 mi/hr. The driver of the car was given a ticket for exceeding the 60 mi/hr speed limit. Why can the officer conclude that the driver exceeded the speed limit? [10 pts]

Solution: Let $P(t)$ be the (unknown) position function of the car, where *t* is measured in hours. The only known values of $P(t)$ are $P(0) = 0$ and $P\left(\frac{28}{60}\right) = 30$. Thus, the average velocity of the car is $\frac{P\left(\frac{28}{60}\right) - P(0)}{\frac{28}{60} - 0} = \frac{30}{\left(\frac{28}{60}\right)} = 60 \cdot \frac{30}{28} > 60$

The Mean-Value theorem guarantees that at some moment of time, the car had to be traveling at the speed of $60 \cdot \frac{30}{28}$ mi/hr.

Extra-Credit

11. State and prove the Mean Value Theorem. [10 pts]

Solution: Consult textbook and/ or your notes.

12. Prove that if $f'(x) = 0$ for all $x \in (a, b)$, then $f(x) = C$ for some constant C. [10 pts]

Solution: Consult textbook and/ or your notes